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A Note on Some New Bounds for Trigonometric Functions Using Infinite Products

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ABSTRACT

In this paper, new sharp bounds for trigonometric functions are established, by using infinite products as main tools. In particular, alternative proofs to well-known results on exponential and polynomial bounds are presented, with improvements.

Keywords: Trigonometric function, Bernoulli inequality, infinite product and exponential bounds.

1. Introduction

Sharp bounds for the basic functions are useful tool in many areas of mathematics, with a wide spectrum of applications. Among others, they allow to establish important properties of sophisticated quantities (monotonicity, convergence...) without a fine analysis. In this study, we aim to provide tight and tractable bounds for $\cos(x)$ and $\sin(x)/x$, which are involved in various integrals, sums and other important quantities (see, for instance, Polyanin and Manzhirov (2008)). Most of the existing bounds for $\cos(x)$ and $\sin(x)/x$ are based on differentiations or integrations or series expansions of well-chosen functions (see, for instance, Mitrinovic (1970)). Here, we focus our attention on the use of infinite products as main tool, providing alternative proofs to some existing results, with improvements.

The rest of the paper is organized as follows. In Section 2, we revisit some exponential bounds for $\cos(x)$ and $\sin(x)/x$ established by Bagul (2017). Then, in Section 3, new polynomial bounds are proved, including an alternative proof for a result established by Bhayo and Sándor (2016). Section 4 is devoted to a near reverse inequality involving $\sin(x)/x$ and $\cos(x/\sqrt{3})$. Wherever it is appropriate, some graphics support the theory. Conclusion is formulated in Section 5.

2. Exponential bounds

2.1 Some well-known results

This section explores exponential bounds for $\cos(x)$ and $\sin(x)/x$. In order to clarify our contributions on the subject, some key results are presented below. The following two theorems have been proved by Bagul (2017). The first result concerns bounds for $\cos(x)$.

Theorem 2.1. *(Bagul, 2017, Theorem 1) For $x \in (0, 1)$, we have*

$$e^{-ax^2} \leqslant \cos(x) \leqslant e^{-x^2/2},$$

with $a \approx 0.6156265$.

The second result is about bounds for $\sin(x)/x$.

Theorem 2.2. *(Bagul, 2017, Theorem 2) For $x \in (0, 1)$, we have*

$$e^{-bx^2} \leqslant \frac{\sin(x)}{x} \leqslant e^{-x^2/6},$$

with $b \approx 0.172604$.

The proofs of (Bagul, 2017, Theorems 1 and 2) are based on the so-called l'Hospital's rule of monotonicity(Anderson et al. (1997)), with the use of non trivial derivative properties of involving functions. In the next subsection, we propose an alternative proofs of these results, with improvements, by the use of infinite series and the Bernoulli inequality presented below.

Proposition 2.1. *[Bernoulli inequality] For* $u, v \in (0,1)$, *we have*

$$1 - uv \geqslant (1-v)^u.$$

We refer to (Shi, 2008, Theorem A) for the general version of the Bernoulli inequality (with less restriction on u and v). An elegant short proof for the considered version is given below.

Proof. For $u, v \in (0,1)$ and $k \geqslant 1$, we have $u^k \leqslant u$. It follows from the logarithmic series expansion that

$$\log(1-uv) = -\sum_{k=1}^{+\infty} \frac{u^k v^k}{k} \geqslant u\left(-\sum_{k=1}^{+\infty} \frac{v^k}{k}\right) = u\log(1-v).$$

Composing by the exponential function, we obtain the desired inequality. □

2.2 Some improvements

The following result is a slight generalization of (Bagul, 2017, Theorem 1), but with a completely different proof.

Proposition 2.2. *For* $\alpha \in (0, \pi/2)$ *and* $x \in (0, \alpha)$, *we have*

$$e^{-\beta x^2} \leqslant \cos(x) \leqslant e^{-x^2/2},$$

with $\beta = [-\log(\cos(\alpha))]/\alpha^2$.

Proof. The proof is centered around the infinite product of the cosine function: for all $x \in \mathbb{R}$, we have

$$\cos(x) = \prod_{k=1}^{+\infty}\left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right).$$

- *Proof of the upper bound.* Using the known inequality : $e^y \geqslant 1 + y$ for $y \in \mathbb{R}$ and $\sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$, for $x \in (0, \pi/2)$ (such that all the terms in the product are strictly positive), we have

$$\begin{aligned}\cos(x) &= \prod_{k=1}^{+\infty} \left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right) \leqslant \prod_{k=1}^{+\infty} \exp\left(-\frac{4x^2}{\pi^2(2k-1)^2}\right) \\ &= \exp\left(-\frac{4x^2}{\pi^2} \sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2}\right) = \exp\left(-\frac{4x^2}{\pi^2} \times \frac{\pi^2}{8}\right) \\ &= e^{-x^2/2}.\end{aligned}$$

- *Proof of the lower bound.* Using the infinite product expression of the cosine function and Proposition 2.1 (with respect to the considered u and v that satisfy $u, v \in (0,1)$), for $x \in (0, \alpha)$, we have

$$\begin{aligned}\cos(x) &= \prod_{k=1}^{+\infty} \left(1 - \frac{4\alpha^2}{\pi^2(2k-1)^2}\frac{x^2}{\alpha^2}\right) \geqslant \prod_{k=1}^{+\infty} \left(1 - \frac{4\alpha^2}{\pi^2(2k-1)^2}\right)^{x^2/\alpha^2} \\ &= \left[\prod_{k=1}^{+\infty} \left(1 - \frac{4\alpha^2}{\pi^2(2k-1)^2}\right)\right]^{x^2/\alpha^2} = (\cos(\alpha))^{x^2/\alpha^2} \\ &= e^{-\beta x^2},\end{aligned}$$

with $\beta = [-\log(\cos(\alpha))]/\alpha^2$.

By combining the obtained upper and lower bounds, we end the proof of Proposition 2.2. □

Note: Taking $\alpha = 1$, we obtain $\beta = -\log(\cos(1)) \approx 0.6156265$, and Proposition 2.2 becomes (Bagul, 2017, Theorem 1).

Proposition 2.3 below gives a generalization of (Bagul, 2017, Theorem 2).

Proposition 2.3. *For $\alpha \in (0, \pi)$ and $x \in (0, \alpha)$, we have*

$$e^{-\gamma x^2} \leqslant \frac{\sin(x)}{x} \leqslant e^{-x^2/6},$$

with $\gamma = [-\log(\sin(\alpha)/\alpha)]/\alpha^2$.

Proof. The proof is centered around the infinite product of the sinc function, i.e. $sinc(x) = \sin(x)/x$ for $x \neq 0$, the so-called Euler formula: for all $x \in \mathbb{R} - \{0\}$, we have

$$\frac{\sin(x)}{x} = \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right).$$

- *Proof of the upper bound.* Using the inequality : $e^y \geqslant 1 + y$ for $y \in \mathbb{R}$ and $\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, for $x \in (0, \pi)$, we have

$$\begin{aligned}\frac{\sin(x)}{x} &= \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right) \leqslant \prod_{k=1}^{+\infty} \exp\left(-\frac{x^2}{\pi^2 k^2}\right) \\ &= \exp\left(-\frac{x^2}{\pi^2} \sum_{k=1}^{+\infty} \frac{1}{k^2}\right) = \exp\left(-\frac{x^2}{\pi^2} \times \frac{\pi^2}{6}\right) \\ &= e^{-x^2/6}.\end{aligned}$$

- *Proof of the lower bound.* Using the infinite product expression of the sinc function and Proposition 2.1, for $x \in (0, \alpha)$, we have

$$\begin{aligned}\frac{\sin(x)}{x} &= \prod_{k=1}^{+\infty} \left(1 - \frac{\alpha^2}{\pi^2 k^2} \frac{x^2}{\alpha^2}\right) \geqslant \prod_{k=1}^{+\infty} \left(1 - \frac{\alpha^2}{\pi^2 k^2}\right)^{x^2/\alpha^2} \\ &= \left[\prod_{k=1}^{+\infty} \left(1 - \frac{\alpha^2}{\pi^2 k^2}\right)\right]^{x^2/\alpha^2} = \left(\frac{\sin(\alpha)}{\alpha}\right)^{x^2/\alpha^2} \\ &= e^{-\gamma x^2},\end{aligned}$$

  with $\gamma = [-\log(\sin(\alpha)/\alpha)]/\alpha^2$. □

Note: Taking $\alpha = 1$, we obtain $\gamma = -\log(\sin(1)) \approx 0.1726037$, and Proposition 2.3 becomes (Bagul, 2017, Theorem 2).

Note: Similar results to Propositions 2.2 and 2.3 can be obtained with hyperbolic functions instead of trigonometric functions. Indeed, the following Bernoulli inequality exists: for $u \in (0, 1)$ and $v \geqslant 0$, we have $1 + uv \geqslant (1 + v)^u$ (see (Shi, 2008, Theorem A)). Therefore, using the same arguments to the proofs of Propositions 2.2 and 2.3 with the infinite products for $\cosh(x)$ and $\sinh(x)/x$, i.e. $\cosh(x) = \prod_{k=1}^{+\infty} \left(1 + \frac{4x^2}{\pi^2(2k-1)^2}\right)$ and $\sinh(x)/x = \prod_{k=1}^{+\infty} \left(1 + \frac{x^2}{\pi^2 k^2}\right)$, we establish that, for $\alpha \geqslant 0$ and $x \in (0, \alpha)$,

- $e^{\theta x^2} \leqslant \cosh(x) \leqslant e^{x^2/2}$, with $\theta = [\log(\cosh(\alpha))]/\alpha^2$, which is appeared in (Bagul, 2018, Remark 2.1) and
- $e^{\zeta x^2} \leqslant \sinh(x)/x \leqslant e^{x^2/6}$, with $\zeta = [\log(\sinh(\alpha)/\alpha)]/\alpha^2$.

3. Polynomial bounds

This section is devoted to the proof of new sharp polynomial bounds or new proofs, for the bounds of $\cos(x)$ and $\sin(x)/x$. As for the previous section, most of the proofs are based on the infinite sums expressions of these functions.

A polynomial lower bound for $\cos(x)$ is given in the result below.

Proposition 3.1. *For $x \in (0, \pi/2)$, we have*

$$\cos(x) \geqslant \left(1 - \frac{4x^2}{\pi^2}\right)^{\pi^2/8}. \tag{1}$$

Proof. The proof combines the infinite product of the cosine function, Proposition 2.1 and $\sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$, but in a different way to the proof of Proposition 2.2.

We have

$$\begin{aligned}\cos(x) &= \prod_{k=1}^{+\infty}\left(1 - \frac{4x^2}{\pi^2}\frac{1}{(2k-1)^2}\right) \geqslant \prod_{k=1}^{+\infty}\left(1 - \frac{4x^2}{\pi^2}\right)^{1/(2k-1)^2} \\ &= \left(1 - \frac{4x^2}{\pi^2}\right)^{\sum_{k=1}^{+\infty} 1/(2k-1)^2} = \left(1 - \frac{4x^2}{\pi^2}\right)^{\pi^2/8}.\end{aligned}$$

□

We provide a graphical illustration of the bounds (1) in Figure 1.

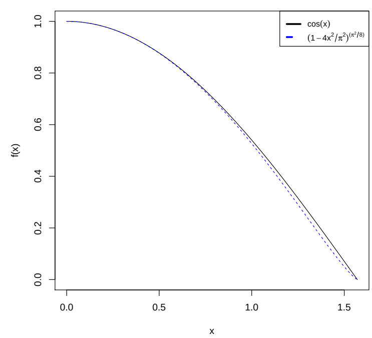

Figure 1: Graphs of the functions in (1) for $x \in (0, \pi/2)$.

Proposition 3.2 below corresponds to (Bhayo and Sándor, 2016, Lower bound in Theorem 1.26). The proof in Bhayo and Sándor (2016) is based on the study of the function $f(x) = \log(x/\sin(x)) - (\pi^2/6)\log(1/(1 - x^2/\pi^2))$. Here we give a more direct proof using infinite products.

Proposition 3.2. *(Bhayo and Sándor, 2016, Lower bound in Theorem 1.26) For $x \in (0, \pi)$, we have*

$$\frac{\sin(x)}{x} \geqslant \left(1 - \frac{x^2}{\pi^2}\right)^{\pi^2/6}. \tag{2}$$

Proof. The proof combines the infinite product of the sinc function, Proposition 2.1 and $\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$:

$$\begin{aligned} \frac{\sin(x)}{x} &= \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\pi^2}\frac{1}{k^2}\right) \geqslant \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\pi^2}\right)^{1/k^2} \\ &= \left(1 - \frac{x^2}{\pi^2}\right)^{\sum_{k=1}^{+\infty} 1/k^2} = \left(1 - \frac{x^2}{\pi^2}\right)^{\pi^2/6}. \end{aligned}$$ □

In the following corollary, we give tight upper bound for $\cos(x)$ based on Proposition 3.2.

Corollary 3.1. *For $x \in (0, \pi)$ we have*

$$\cos(x) \leqslant 1 + \frac{3\pi^2}{\pi^2+6}\left[\left(1-\frac{x^2}{\pi^2}\right)^{(\pi^2+6)/6} - 1\right]. \tag{3}$$

Proof. Using (2) and integrating we have

$$\int_0^x \sin(t)dt \geqslant \int_0^x \left(1-\frac{t^2}{\pi^2}\right)^{\pi^2/6} tdt.$$

After elementary calculus, we obtain

$$1-\cos(x) \geqslant -\frac{3\pi^2}{\pi^2+6}\left[\left(1-\frac{x^2}{\pi^2}\right)^{(\pi^2+6)/6} - 1\right],$$

implying that

$$\cos(x) \leqslant 1 + \frac{3\pi^2}{\pi^2+6}\left[\left(1-\frac{x^2}{\pi^2}\right)^{(\pi^2+6)/6} - 1\right].$$

This ends the proof of Corollary 3.1. □

The upper bound of Corollary 3.1 can be seen in Figure 2.

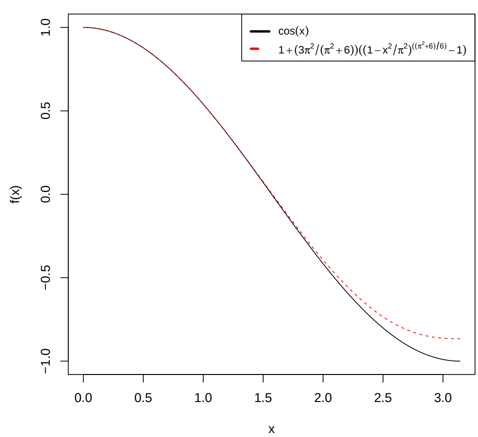

Figure 2: Graphs of the functions in (3) for $x \in (0, \pi)$.

Note: Similar results to Propositions 3.1 and 3.2 can be obtained with hyperbolic functions instead of trigonometric functions. Indeed, using the following Bernoulli inequality: for $u \in (0,1)$ and $v \geqslant 0$, we have $1 + uv \geqslant (1+v)^u$ and the infinite products for $\cosh(x)$ and $\sinh(x)/x$, we prove that, for $x \geqslant 0$,

- $\cosh(x) \geqslant \left(1 + 4x^2/\pi^2\right)^{\pi^2/8}$,
- $\sinh(x)/x \geqslant \left(1 + x^2/\pi^2\right)^{\pi^2/6}$.

4. On a "near reverse" inequality

This section is devoted to an upper bound for $\sin(x)/x$. It is proved in Mitrinovic (1970) that, for $x \in (0, \pi\sqrt{3}/2)$, we have

$$\frac{\sin(x)}{x} \geqslant \cos\left(\frac{x}{\sqrt{3}}\right).$$

This inequality is known to be very sharp. The next result determines a "near reverse" inequality using infinite products techniques in the proof.

Proposition 4.1. *For $x \in (0, \pi\sqrt{3}/2)$, we have the inequality*

$$\cos\left(\frac{x}{\sqrt{3}}\right) f(x) \geqslant \frac{\sin(x)}{x}, \tag{4}$$

where

$$f(x) = \frac{\pi^2 - x^2}{\pi^2 - (4/3)x^2}.$$

Proof. By using the infinite product for the cosine function, for $x \in (0, \pi\sqrt{3}/2)$,

$$\begin{aligned}\cos\left(\frac{x}{\sqrt{3}}\right) &= \prod_{k=1}^{+\infty}\left(1 - \frac{4x^2}{3\pi^2(2k-1)^2}\right)\\ &= \left(1 - \frac{4x^2}{3\pi^2}\right)\prod_{k=2}^{+\infty}\left(1 - \frac{4x^2}{3\pi^2(2k-1)^2}\right).\end{aligned}$$

Now observe that, for $k \geqslant 2$, we have $(3/4)(2k-1)^2 - k^2 = (1/4)(2k^2 - 3) \geqslant 0$.

Therefore, for $k \geqslant 2$, we have $1 - x^2/(\pi^2 k^2) \leqslant 1 - 4x^2/\left[3\pi^2(2k-1)^2\right]$ and

$$\cos\left(\frac{x}{\sqrt{3}}\right) \geqslant \left(1 - \frac{4x^2}{3\pi^2}\right)\prod_{k=2}^{+\infty}\left(1 - \frac{x^2}{\pi^2 k^2}\right)$$
$$= \left(\frac{1 - 4x^2/(3\pi^2)}{1 - x^2/\pi^2}\right)\prod_{k=1}^{+\infty}\left(1 - \frac{x^2}{\pi^2 k^2}\right) = \frac{1}{f(x)}\frac{\sin(x)}{x}.$$

This implies the desired inequality since $f(x) > 0$ for any $x \in (0, \pi\sqrt{3}/2)$, ending the proof of Proposition 4.1. □

Notes: In fact, we have $f(x) > 1$ with $f(x)$ very close to 1 for x not is in the neighborhood of $\pi\sqrt{3}/2$.

Figure 3 illustrates the obtained bounds.

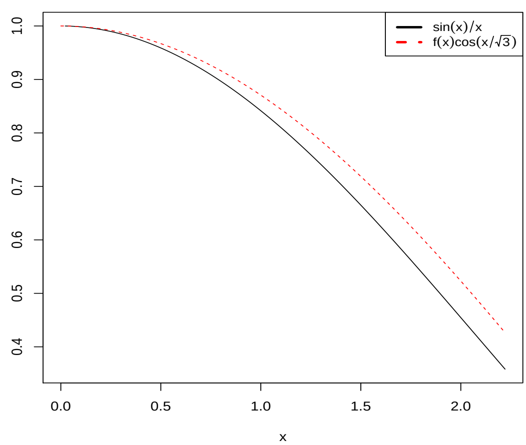

Figure 3: Graphs of the functions in (4) for$x \in (0, \pi\sqrt{2}/2)$.

5. Conclusion

In this paper, we have proved sharp bounds for trigonometric functions through the use of infinite products, which remains an original approach as far as we know. As main contributions, several existing bounds are improved, relaxing some assumptions on the possible values of the variable, and related new results are proved. In particular, a near reverse sharp version of an inequality established by Mitrinovic (1970) is proved. The findings are supported by graphical illustrations. We believe that the methodology based on infinite products opens some new perspectives for the refinement of known sharp bounds, beyond the those involving trigonometric functions.

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